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CARDINALITY AND NILPOTENCY OF LOCALIZATIONS OF GROUPS AND G-MODULES

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ABSTRACT

We consider the effect of a coaugmented idempotent functor J in the the category of groups or G-modules where G is a fixed group. We are interested in the 'extent' to which such functors change the structure of the objects to which they are applied. Some positive results are obtained and examples are given concerning the cardinality and structure of $J(A)$ in terms of the cardinality and structure of A, where the latter is a torsion abelian group. For non-abelian groups some partial results and examples are given connecting the nilpotency classes and the varieties of a group G and $J(G)$. Similar but stronger results are obtained in the category of G-modules.

Introduction

This paper deals with the behavior of localization functors in the categories of groups, abelian groups and G-modules for a fixed group G. Nevertheless, most of the results apply to coaugmented idempotent functors. The main interest is in the extent to which such functors preserve the algebraic or even the underlying set structure of the objects to which they are applied. It is shown (see 2.8) that the cardinality of an abelian torsion group cannot increase arbitrarily after localization. For a general group the group structure may vastly change after localization (see 3.4) but for certain nilpotent groups the opposite happens (3.3). Similar results hold for G-modules (4.2 and 4.5).

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1. Idempotent coaugmented functors

Let $\mathcal{C}$ be a category. A coaugmented functor F is a functor $F: \mathcal{C} \to \mathcal{C}$ together with a natural transformation a: Id $\to F$ called coaugmentation. The coaugmentation is called idempotent if $a_{FX}, F(a_X): FX \to FFX$ are equal and are an isomorphism. Given a coaugmented idempotent functor F, a **local** object is an object T in $\mathcal{C}$ isomorphic with FX for some $X \in \mathcal{C}$. An F-**equivalence** is a morphism $f: X \to Y$ in $\mathcal{C}$ which becomes an isomorphism after applying F. Clearly the coaugmentation is always an F-equivalence $a_X: X \to FX$. From these definitions it follows that for every $X \in \mathcal{C}$ and every local object T there is a bijection $a^*: \mathrm{Hom}_{\mathcal{C}}(FX, T) \to \mathrm{Hom}_{\mathcal{C}}(X, T)$. This is referred to as the universality of a. Coaugmented idempotent functors will be denoted CIF's.

Very frequent examples of CIF's are localization functors with respect to maps. Given a morphism $f: A \to B$ in $\mathcal{C}$, we define an object T to be f-local if $f^*: \mathrm{Hom}_{\mathcal{C}}(B, T) \to \mathrm{Hom}_{\mathcal{C}}(A, T)$ is a bijection. A localization functor $L_f: \mathcal{C} \to \mathcal{C}$ (if it exists) is a CIF such that $L_f(X)$ is f-local for all $X \in \mathcal{C}$. For a more extensive discussion one may consult [3] or [1].

The results of this article were motivated by these functors but apply to all CIF. With this in mind, throughout the article (L, a) denotes a CIF. It is interesting to note that it is still unknown whether all CIF's in the category of groups have the form L_f for some f.

Given (L, a), the following are immediate consequences of the definitions, hence only indications of their proofs will be given (if any).

1.1. If $\varphi: X \to Y$ is a morphism in $\mathcal{C}$ where Y is local and φ is an L-equivalence, then $Y \cong LX$.

1.2. Assume $\varphi: X \to T$ is a morphism such that T is local and for every local object Y there is a bijection $\varphi^*: \mathrm{Hom}_{\mathcal{C}}(T, Y) \to \mathrm{Hom}_{\mathcal{C}}(X, Y)$. Then $T \cong LX$ and φ represents the coaugmentation.

1.3. Every retract of a local object is a local object.

1.4. An object T is local if and only if $a^*: \mathrm{Hom}(LX, T) \to \mathrm{Hom}(X, T)$ is bijective for all $X \in \mathcal{C}$.

1.5. Every inverse limit of local objects is a local object.

1.6. For a diagram $\{X_i\}_{i\in I}$, there is a natural isomorphism

$$L(\varinjlim X_i) \to L(\varinjlim LX_i).$$

Proof: Let $\mathcal{E}$ be the subcategory of local objects in $\mathcal{C}$. Note that L is left adjoint to the inclusion functor $i\colon \mathcal{E} \to \mathcal{C}$ and that the direct limit in the subcategory $\mathcal{E}$ of local objects is given by $\varinjlim_{\mathcal{E}}\{X_i\} = L(\varinjlim_{\mathcal{C}}\{X_i\})$. ∎

Since throughout the article localization functors are used, it seems in place to remark on the existence of such functors. In fact, the category $\mathcal{C}$ has to satisfy mild conditions in order that the functors L_f exist for all morphisms f in $\mathcal{C}$. Assume that $\mathcal{C}$ satisfies the following:

(a) All limits and colimits exist. This implies that given a diagram $\{X_i\}_{i\in I}$ of objects in $\mathcal{C}$, there is a bijection $\mathrm{Hom}_{\mathcal{C}}(\varinjlim_i X_i, Y) \cong \varprojlim_i \mathrm{Hom}_{\mathcal{C}}(X_i, Y)$ for all $Y \in \mathcal{C}$ (cf. [11], Theorem 2.7.2).

(b) An object $C \in \mathcal{C}$ is said to be λ-**small** if λ is an ordinal and for every λ-diagram $\{X_\alpha\}_{\alpha<\lambda}$ the natural map $\varinjlim \mathrm{Hom}_{\mathcal{C}}(C, X_\alpha) \to \mathrm{Hom}_{\mathcal{C}}(C, \varinjlim X_\alpha)$ is a bijection. We assume that for every two objects $A, B \in \mathcal{C}$ there exists an ordinal λ such that both A and B are λ-small.

Under these assumptions, an argument parallel to the one given in [5] chapter 1.B may be applied to construct L_f for any f in the category $\mathcal{C}$. For an object $X \in \mathcal{C}$ and a *set* S, one should define $X \times S := \coprod_{s\in S} X$. Then there is an adjunction $\mathrm{Hom}_{\mathcal{C}}(X \times S, Y) \cong \mathrm{Hom}_{\mathcal{S}\mathrm{ets}}(S, \mathrm{Hom}_{\mathcal{C}}(X, Y))$ for all $X, Y \in \mathcal{C}$. Now one should follow the construction given in [5] chapter 1.B, replacing all homotopy limits and colimits by limits and colimits and understanding expressions in the form $A \times X^B = A \times \mathrm{Hom}_{\mathcal{C}}(B, X)$ as above. A detailed construction can be found in [9].

Finally, notice that the categories of (abelian) groups and G-modules for a fixed group G all satisfy conditions (a) and (b), hence localization functors exist there. We thus freely use localization functors in these three categories.

For the following, $\mathcal{C}$ denotes one of the categories of groups, abelian groups or G-modules. Thus we feel free to use the underlying set structure of the objects in $\mathcal{C}$ and hence talk about 'images' of maps etc. Throughout L is a CIF and a is its coaugmentation.

LEMMA 1.7: *Let G be an object of $\mathcal{C}$ and $a\colon G \to LG$ be the coaugmentation. Assume that T is a local subobject of LG. If $\mathrm{im}(a) \subseteq T$ then $T = LG$.*

Proof: Factor a through T, i.e. write $a = j\bar{a}$ where $j\colon T \to LG$ is the inclusion map and $\bar{a}\colon G \to T$ is the restriction of the range of a to T. By universality of

LG, there exists a map φ rendering the following diagram commutative.

$$\begin{array}{ccccc} G & \xrightarrow{\bar{a}} & T & \xrightarrow{j} & LG \\ {\scriptstyle a}\downarrow & \nearrow{\scriptstyle \varphi} & & & \\ LG & & & & \end{array}$$

However $a = j\bar{a} = j\varphi a$ implies $j\varphi = 1_{LG}$, in particular j is an epimorphism, hence an isomorphism. ∎

1.8. Every direct factor of a local object in $\mathcal{C}$ is local. Further, if $G \in \mathcal{C}$ and $\mathrm{im}(a) \subseteq S$ where S is a direct factor of LG, then $S = LG$.

1.9. Let $G_1, \dots, G_n$ be G-modules (or abelian groups) and let $a_i\colon G_i \to LG_i$ be their coaugmentation maps. Then $L(G_1 \oplus \cdots \oplus G_n) \cong LG_1 \oplus \cdots \oplus LG_n$ and under this isomorphism $a_{G_1 \oplus \cdots \oplus G_n} = a_{G_1} \oplus \cdots \oplus a_{G_n}$.

1.10. If $\varphi\colon T \to S$ is a homomorphism of local objects then $\ker \varphi$ is also local.

1.11. If G is a subobject of T where T is local, then the coaugmentation $a\colon G \to LG$ is a monomorphism.

Proofs: For the first assertion of 1.8 use 1.3. For the second assertion use the first one together with 1.7.

For the proof of 1.9 note that $LG_1 \times \cdots \times LG_n$ is local. Therefore it is enough to show that $G_1 \times \cdots \times G_n \xrightarrow{a_1 \times \cdots \times a_n} LG_1 \times \cdots \times LG_n$ satisfies the universal property with respect to all local objects. Indeed for every local object T we have

$$\begin{array}{ccc} \mathrm{Hom}(LG_1 \oplus \cdots \oplus LG_n, T) & \xrightarrow{\cong} & \prod_i \mathrm{Hom}(LG_i, T) \\ {\scriptstyle (a_1 \oplus \cdots \oplus a_n)^*}\downarrow & & \downarrow{\scriptstyle \cong} \\ \mathrm{Hom}(G_1 \oplus \cdots \oplus G_n, T) & \xrightarrow[\cong]{} & \prod_i \mathrm{Hom}(G_i, T). \end{array}$$

For 1.10 use 1.5 and the diagram $\{1 \to T \xleftarrow{\varphi} S\}$ where 1 denotes the trivial group or G-module. Note that always $L(1) = 1$. For the proof of 1.11, note that the inclusion $j\colon G \to T$ has a factorization $G \xrightarrow{a} LG \to T$. ∎

2. Localization of abelian groups

In this section (L, a) denotes a CIF in the category $\mathcal{A}b$ of abelian groups. In this section, unless otherwise stated, by the word group we mean an abelian group. It will be shown (Theorem 2.3) that if G is a bounded group (cf. [6]) then LG is a quotient group of G. A similar result applies for all divisible groups (Lemma 2.1).

The latter fact is the key to Theorem 2.4 stating that if G is a reduced group (cf. [6]) then LG is also reduced and furthermore $|LG| \le |G|^{\aleph_0}$. In particular it follows (Corollary 2.8) that if G is a torsion group then $|LG| \le |G|^{\aleph_0}$.

Recall that every abelian group G is a direct sum $G = D \oplus R$ where D is divisible and R is reduced (cf. [6], sect. 21). In the light of 1.9, we may consider separately the case of localization of divisible groups and the case of localization of reduced groups.

LEMMA 2.1: *Let D be a divisible abelian group. Then $a\colon D \to LD$ is an epimorphism.*

Proof: Since D is divisible so is $\mathrm{im}(a)$, and therefore it is a direct summand in LD. But by 1.8 it follows that $\mathrm{im}(a) = LD$. ∎

In view of this lemma it is now obvious that divisibility is preserved under localization. Another useful consequence is that

$$L(\mathbb{Z}(p^\infty)) = \begin{cases} \mathbb{Z}(p^\infty) & \text{if } \mathbb{Z}(p^\infty) \text{ is local} \\ 0 & \text{otherwise} \end{cases}$$

which follows from the fact that the quasicyclic groups $\mathbb{Z}(p^\infty)$ (cf. [6]) are divisible and their quotients are either 0 or isomorphic with themselves.

PROPOSITION 2.2: *Let $G = \bigoplus_{\alpha<\lambda} D_\alpha$ where D_α are local divisible groups. Then G is a local group.*

Proof: Clearly G is divisible, i.e. an injective object in the abelian category $\mathcal{A}$b, thus it is a direct summand of $\prod_{\alpha<\lambda} D_\alpha$. The result follows from 1.5 and 1.8. ∎

THEOREM 2.3: *Let G be a bounded abelian group. Then $a\colon G \to LG$ is an epimorphism.*

Proof: Note that every bounded abelian group is a finite direct sum of its p-components, so it suffices by 1.9 to prove the result for bounded abelian p-groups.

Let, then, G be a bounded abelian p-group and let C be the cokernel in $G \xrightarrow{a} LG \to C$. Clearly LG and C are bounded abelian p-groups since the exponent of G annihilates LG. Moreover, 1.6 implies that $LC = 0$. Likewise, L kills all quotients of C.

Now we prove that $C = 0$. Assume to the contrary that $C \ne 0$. Then clearly $LG \ne 0$. Therefore LG contains a subgroup isomorphic with $\mathbb{Z}/p$. From 1.1 it follows that $L(\mathbb{Z}/p) \ne 0$. On the other hand, since we assume $C \ne 0$, $\mathbb{Z}/p$ is clearly a quotient of C, hence it is killed by L, a contradiction. ∎

We now come to the main result of this section. It gives an answer to the question concerning the cardinality of the localization of a torsion group.

THEOREM 2.4: *Let G be a reduced abelian torsion group. Then LG is a reduced abelian group and $|LG| \leq |G|^{\aleph_0}$.*

LEMMA 2.5: *Let K be an abelian group and p a prime. Assume $\mathbb{Z}/p \subseteq K$ and $\mathrm{Hom}(H,K) = 0$ for some group H. Then H is p-divisible.*

Proof: Assume to the contrary that $pH \neq H$. The composite $H \to H/pH \cong \bigoplus \mathbb{Z}/p \to \mathbb{Z}/p \subseteq K$ is a non-trivial homomorphism $H \to K$, contradicting $\mathrm{Hom}(H,K) = 0$. ∎

LEMMA 2.6: *Let G be a reduced abelian group and let $H \leq G$ such that G/H is divisible. Then $|G| \leq |H|^{\aleph_0}$.*

Proof: Compare [6] sect. 34, exercise 9. A detailed proof may be found in [9]. ∎

PROPOSITION 2.7: *If $\mathbb{Z}(p^\infty)$ is local, then every abelian p-group is local.*

Proof: Let G be an abelian p-group. Let E be its divisible-hull (cf. [6], sect. 24). Clearly $E \cong \bigoplus_\lambda \mathbb{Z}(p^\infty)$ and similarly E/G. By Proposition 2.2 both E and E/G are local. Note that $G = \ker(E \to E/G)$ and use 1.10 to conclude that G is local. ∎

Proof of Theorem 2.4: Let G be an abelian reduced torsion group and let G_p be its p-component. We have a direct sum representation $G = \bigoplus_p G_p$. We may assume that if $L(G_q) = 0$ for some prime q, then $G_q = 0$ because $L(\bigoplus_p G_p) \cong L(G_q) \oplus L(\bigoplus_{p\neq q} G_p) \cong L(\bigoplus_{p\neq q} G_p)$.

Let P be the set of primes for which $\mathbb{Z}(p^\infty)$ is local, and let Q be the set of primes for which $\mathbb{Z}(p^\infty)$ is not local, in particular $L(\mathbb{Z}(p^\infty)) = 0$ for all $p \in Q$. Now define $T = \bigoplus_{p\in P} G_p$, $S = \bigoplus_{q\in Q} G_q$; then $G = T \oplus S$.

CLAIM: *T is local.*

Proof: Let E be the divisible-hull of T. Then $E \cong \bigoplus_{p\in P}(\bigoplus_{\lambda_p} \mathbb{Z}(p^\infty))$, $T \subseteq E$ and E/T is divisible. Clearly $E/T = \bigoplus_{p\in P}(\bigoplus_{\kappa_p} \mathbb{Z}(p^\infty))$. Proposition 2.1 implies that E and E/T are local because both groups are direct sums of the local groups $\mathbb{Z}(p^\infty)$. The claim follows by 1.10.

CLAIM: *LS is reduced.*

Proof: Let $a\colon S \to LS$ be the coaugmentation map, and assume LS is not reduced. Represent $LS = D \oplus R$ where D is divisible and R is reduced. Note that

D and R are local. For every $p \in P$ multiplication by p is an automorphism of S, therefore it is an automorphism of LS, and it follows that $\mathbb{Z}(p^\infty) \not\subseteq LS$. For every $q \in Q$, $\mathbb{Z}(q^\infty)$ is not local by assumption and in particular $\mathbb{Z}(q^\infty) \not\subseteq LS$. It follows that D is torsion-free, i.e. $D \cong \bigoplus_\lambda \mathbb{Q}$. It is now clear that $\mathrm{im}(a) \subseteq R$ and by Proposition 1.11, $LS = R$, i.e. LS is reduced as the claim states.

Notice that $LG \cong LS \oplus LT \cong LS \oplus T$ where LS is reduced. However T is also reduced, being a direct summand of G, which is reduced by assumption. This proves the first assertion of the theorem.

Let $H = \mathrm{im}(a) \subseteq LG$. We now show that LG/H is divisible. It suffices to show that LG/H is p-divisible for every prime p. Let p be a prime and G_p the p-component of G. If $G_p = 0$ then G is uniquely p-divisible as it is a torsion group. This implies that LG is also uniquely p-divisible, and therefore LG/H is p-divisible. Now consider the case $G_p \neq 0$. By assumption $L(G_p) \neq 0$ and therefore the homomorphism $a_{G_p}\colon G_p \to L(G_p)$ is not trivial. In particular $\mathrm{im}(a) \neq 0$ and it follows that $L(G_p)$ contains a subgroup of order p. Furthermore, LG contains such a subgroup because $LG = L(G_p \oplus U) \cong L(G_p) \oplus L(U)$ for a suitable U. Using 1.6 we conclude that $\mathrm{Hom}(LG/H, LG) \cong \mathrm{Hom}(L(\mathrm{coker}\, a), LG) = 0$ and, by Lemma 2.5, LG/H is p-divisible. Now use Lemma 2.6 to conclude that $|LG| \leq |H|^{\aleph_0} \leq |G|^{\aleph_0}$. ∎

COROLLARY 2.8: *Let G be an abelian torsion group. Then $|LG| \leq |G|^{\aleph_0}$.*

Proof: Represent $G = D \oplus R$ where D is divisible and R is reduced. The corollary follows from Theorem 2.4, Lemma 2.1 and Proposition 1.9. ∎

One may be tempted to believe that in Theorem 2.4 the assumption that G is torsion implies that LG is also a torsion group. This is however false as is shown in the following examples.

Example 2.9: Let $p_1, p_2, \ldots$ be an ordering of the positive primes. Let $S = \bigoplus_i \mathbb{Z}/p_i$ and let $P = \prod_i \mathbb{Z}/p_i$. It is not hard to check that S is reduced and that P/S is divisible, hence the inclusion $j\colon S \to P$ induces a bijection

$$j^*\colon \mathrm{Hom}(P,P) \to \mathrm{Hom}(S,P).$$

Thus, P is not torsion and $L_j(S) \cong P$. In fact, $P = \mathrm{Ext}(\mathbb{Q}/\mathbb{Z}, S)$ where $\mathrm{Ext}(\mathbb{Q}/\mathbb{Z}, -)$ is the "total-Ext-completion" which is a CIF ([12] or [2] p. 165).

Even if G is a p-group, LG need not be one. To show this choose a prime p and take $G = \bigoplus_{i \geq 1} \mathbb{Z}/p^i$. The Ext-completion functor $\mathrm{Ext}(\mathbb{Z}(p^\infty), -)$ is a CIF (see [2], pp. 165, 166 and 172). The natural map $f\colon G \to \mathrm{Ext}(\mathbb{Z}(p^\infty), G)$ satisfies

$L_f(G) \cong \mathrm{Ext}(\mathbb{Z}(p^\infty), G)$ and example 4.2(ii) on page 180 in [2] shows that the latter group is not a torsion group.

We remark that torsion free groups behave very differently. For example, $j: \mathbb{Z} \to \mathbb{Q}$ is a localizing map where $\mathbb{Z}$ is reduced but $\mathbb{Q}$ is divisible. As shown in [4] one cannot bound $|L\mathbb{Z}|$.

3. Nilpotency and words in the category of groups

We show in Theorem 3.3 that localization preserves nilpotency class 2. This was first proved by W. G. Dwyer and E. D. Farjoun (unpublished). The result is sharpened by showing that in fact, if G is nilpotent of class 2 then LG lies in the variety defined by G (cf. [10]), namely every law in G is also a law in LG. Note that in particular it follows that applying a CIF to an abelian group yields an abelian group. No generalization of the above result is known for arbitrary nilpotent groups.

Let H be a group. We shall denote the center of H by $Z(H)$. The upper central series of H is denoted by $Z_i(H)$ where $Z_0(H) = 1$ and $Z_{i+1}(H)/Z_i(H) = Z(H/Z_i(H))$. If H is nilpotent, $c(H)$ denotes its nilpotency class.

There is a map $\mu_H: H \times Z(H) \to H$ defined by $\mu_H(h,z) = hz$, which is evidently a group homomorphism since z is in the center.

LEMMA 3.1: *Let G be local. Then $Z(G)$ and $Z_2(G)$ are local.*

Proof: First, note that $Z(G)$ is the inverse limit of the diagram consisting of the object G alone, and with one arrow for each inner automorphism of G. Then 1.5 shows that $Z(G)$ is local.

For every $x \in G$ define a homomorphism $\phi_x: G \times Z(G) \to G$ by $\phi_x(g,z) = \mu_G(g^x, z) = g^x z = x^{-1}gx \cdot z$. To see that $Z_2(G)$ is local, observe that it is the pullback in the following diagram, in which j is the inclusion, $\Delta(g) = \prod_{x\in G}(g,g)$ and $\epsilon(z) = \prod_{x\in G}(z,[x,z])$.

$$\begin{array}{ccc} Z_2(G) & \xrightarrow{\quad j \quad} & G \\ \Big\downarrow{\scriptstyle \epsilon} & & \Big\downarrow{\scriptstyle \Delta} \\ \prod_{x\in G}(G \times Z(G)) & \xrightarrow[\prod_{x\in G}(1_G\times\phi_x)]{} & \prod_{x\in G}(G\times G). \end{array}$$ ∎

PROPOSITION 3.2: *Let G be a group and let $a: G \to LG$ be the coaugmentation. Then $a(Z_2(G)) \subseteq Z_2(LG)$.*

Proof: For $w \in Z_2(G)$ define a group homomorphism $\varphi_w\colon G \to Z(G)$ by $\varphi_w(x) = [x, w]$. We obtain a commutative square in which $\bar{\varphi}_w$ extends $a|_{Z(G)} \circ \varphi_w$:

$$\begin{array}{ccc} G & \xrightarrow{\varphi_w} & Z(G) \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle a|_{Z(G)}} \\ LG & \dashrightarrow_{\bar{\varphi}_w} & Z(LG). \end{array}$$

Let us compute $\bar{\varphi}_w$. There is a commutative diagram:

$$\begin{array}{ccccccc} G & \xrightarrow{\Delta} & G \times G & \xrightarrow{1_G \times \varphi_w} & G \times Z(G) & \xrightarrow{\mu_G} & G \\ {\scriptstyle a}\downarrow & & {\scriptstyle a\times a}\downarrow & & \downarrow{\scriptstyle a \times a|_{Z(G)}} & & \downarrow{\scriptstyle a} \\ LG & \xrightarrow[\Delta]{} & LG \times LG & \xrightarrow[1_{LG} \times \bar{\varphi}_w]{} & LG \times Z(LG) & \xrightarrow[\mu_{LG}]{} & LG \end{array}$$

where the composite in the top row is precisely the map $g \mapsto w^{-1}gw$. The composite in the bottom row is the map $x \mapsto x\bar{\varphi}_w(x)$. Defining $\Psi\colon LG \to LG$ by the formula $\Psi(x) = x^{a(w)}$ renders the outer rectangle in the above diagram commutative, hence the composite in the bottom row equals Ψ. This yields $\bar{\varphi}_w(x) = [x, a(w)]$, and in particular $[LG, w] \subseteq Z(LG)$. As $w \in Z_2(G)$ was arbitrary we conclude that $[LG, a(Z_2(G))] \subseteq Z(LG)$, i.e. $a(Z_2(G)) \subseteq Z_2(LG)$. ∎

THEOREM 3.3: *Let G be nilpotent of class at most 2. Then LG is nilpotent of class at most 2. Furthermore, if $w(x_1, \dots, x_n)$ is a word which is satisfied in G, namely the verbal subgroup $w(G)$ vanishes (see [10]), then w is satisfied in LG.*

Proof: By Lemma 3.1, $Z_2(LG)$ is local and by Lemma 3.2, $a(G) = a(Z_2(G)) \subseteq Z_2(LG)$. Use Lemma 1.7 to conclude that $Z_2(LG) = LG$.

For the second assertion of the theorem assume first that the word w has the form $w(x) = x^p$ for some integer p. There exists an integer k_p such that

$$(xy)^p = x^p y^p [y, x]^{k_p} \mod \Gamma^3 G.$$

This follows from $(xy)^p \equiv x^p y^p \pmod{\Gamma^2 G}$ and due to 10.2.3 in [7] (alternatively use the results of chapter 11 there).

From the assumption that w is satisfied in G, we have $(xy)^p = x^p y^p = 1$ for all $x, y \in G$, hence $[y, x]^{k_p} = 1$ for all $x, y \in G$. Since $c(LG) \leq 2$ for every $x \in LG$, there is a homomorphism $\varphi_x(y) = [y, x]^{k_p}$. If $x \in \mathrm{im}(a)$ then $\varphi_x \circ a$ is trivial, and therefore φ_x is trivial. It follows that $[LG, \mathrm{im}(a)]^{k_p} = 1$. Now, given $y \in LG$ we define ψ_y: $LG \to Z(LG)$ by the formula $\psi_y(x) = [y,x]^{k_p}$. Clearly $\varphi_y \circ a$ is trivial, and this shows that $[LG, LG]^{k_p} = 1$. Now, since $c(LG) \leq 2$,

$$(xy)^p = x^p y^p [y,x]^{k_p} = x^p y^p \quad (x, y \in LG).$$

It follows that $\psi : x \mapsto x^p$ is a homomorphism in LG. Since $\psi \circ a$ is trivial, ψ is trivial and the theorem is proved in the case $w(x) = x^p$.

Consider now the general case. Note that given a word $w(x_1, \ldots, x_n)$ it is equivalent to a word

$$x_1^{e_1} \cdots x_n^{e_n} \cdot \prod_{i<j\leq n} c_{i,j}^{f_{i,j}} \cdot T$$

where T is a product of commutators of length greater than 2, $c_{i,j} = [x_i, x_j]$ and $f_{i,j}$ are integers. Observe that T vanishes in G and LG as $c(G), c(LG) \leq 2$. Choose $1 \leq i \leq n$ and set $x_j = 1$ for all $j \neq i$. We see that $x_i^{e_i}$ holds in G and therefore in LG. It follows that the word $\prod_{i<j} c_{i,j}^{f_{i,j}}$ holds in G and we are led to show that it holds in LG. Let i, j be integers such that $f_{i,j} \neq 0$. By setting $x_\ell = 1$ for all $\ell \neq i, j$ we see that $[x_i, x_j]^{f_{i,j}} = 1$ in G and we attempt to show that this is the case in LG. Given $x \in a(G)$ the map $y \mapsto [y,x]^{f_{i,j}}$ is a homomorphism $LG \to Z(LG)$ which vanishes on $a(G)$ and consequently on LG, namely $[a(G), LG]^{f_{i,j}} = 1$. Now apply the same argument for the map $x \mapsto [x,y]^{f_{i,j}}$ where $y \in LG$ to prove that $[LG, LG]^{f_{i,j}} = 1$, as desired. It follows that this homomorphism is trivial in LG, as desired. ∎

It is not known, in general, whether LG is nilpotent whenever G is. Notice that the proof of Theorem 3.3 follows the pattern:

1. Show that if T is a local group then $Z_n(T)$ are local groups ($n \geq 1$).
2. Show that if a: $G \to LG$ is the coaugmentation then $a(Z_n(G)) \subseteq Z_n(LG)$.
3. Conclude that if G is nilpotent of class n then LG is nilpotent of class $\leq n$ by using Lemma 1.7.

No generalization of the proofs given for the cases $n = 1, 2$ is known. It is noted that if G is nilpotent and it is known that LG is nilpotent then $c(LG) \leq c(G)$ (see next theorem). This gives evidence to the credibility of the above scheme of proving that LG is nilpotent whenever G is.

As for the word problem, let the previous theorem not mislead. The next example shows that in general LG need not lie in the variety defined by G. In this example the groups are not nilpotent, and so we still do not know if this is true when G (or even LG) are nilpotent.

Example 3.4: Let S_n and A_n denote the Symmetric and Alternating groups on n letters respectively. The standard inclusions $j\colon S_n \to S_{n+1}$ and $k\colon A_n \to A_{n+1}$ for $n > 6$ satisfy $L_j(S_n) \cong S_{n+1}$ and $L_k(A_n) \cong A_{n+1}$. In particular, in S_{10} the word $w(x) = x^{10!}$ holds, but it does not hold in S_{11}. Moreover, $S_{11} = L_j(S_{10})$ contains elements of order 11 although S_{10} does not.

Consider $j\colon S_n \to S_{n+1}$. We show that every map $\varphi\colon S_n \to S_{n+1}$ extends uniquely to a map $S_{n+1} \to S_{n+1}$. Observe that every $\varphi\colon S_n \to S_{n+1}$ is either trivial, has A_n as its kernel or is an embedding. If φ is trivial then any extension must have S_n in its kernel, and hence is trivial. If $\ker\varphi = A_n$ then there exists a unique isomorphism $\mathrm{im}(\varphi) \cong C_2$ and the composite $S_{n+1} \to S_{n+1}/A_{n+1} \cong C_2 \cong \mathrm{im}(\varphi)$ is a unique extension for φ. Finally, consider the case φ is injective. Let $H = \mathrm{im}(\varphi)$. The action of S_{n+1} on the (say, right) cosets of H gives rise to an automorphism $\theta\colon S_{n+1} \to S_{n+1}$ which sends H to the standard copy of S_n in S_{n+1}. Thus, the composite $\theta \circ \varphi$ may be considered as an automorphism of S_n.

Clearly, showing that φ has a unique extension is equivalent to showing that the composite $\theta\circ\varphi$ has a unique extension to S_{n+1}. The assumption $n > 6$ implies that S_n is complete (see [13], Theorem 7.3). Therefore there exists $\alpha \in S_n$ such that $\theta \circ \varphi = \tau_\alpha \in \mathrm{Inn}\,(S_n)$. It follows that $\tau_\alpha \in \mathrm{Inn}\,(S_{n+1})$ is an extension of $\theta \circ \varphi$. If ρ is another extension, then $|\mathrm{im}(\rho)| > 2$ implies that $\rho \in \mathrm{Aut}\,(S_{n+1})$. Completeness of S_{n+1} implies that $\rho = \tau_\beta \in \mathrm{Inn}\,(S_{n+1})$ for some $\beta \in S_{n+1}$. It now follows that $\tau_{\alpha\beta^{-1}} \in \mathrm{Inn}\,(S_{n+1})$ induces the identity on S_n, namely $\alpha\beta^{-1} \in C_{S_{n+1}}(S_n) = 1$. Therefore $\rho = \tau_\beta = \tau_\alpha$ as desired.

The proof that $k\colon A_n \to A_{n+1}$ localizes is similar. We only note that all automorphisms of A_n are induced by conjugation in S_n (note that $n > 6$).

THEOREM 3.5: *Let $a\colon G \to LG$ be the coaugmentation and assume G is nilpotent of class c. If LG is nilpotent then $c(LG) \le c(G)$. In fact $c(LG) = c(\mathrm{im}(a))$.*

Recall that for all $1 \le i \le c(G)$, $\Gamma^i G \subseteq Z_{c-i+1}(G)$ (cf. [7] 10.2.2) where $\Gamma^i G$ denotes the lower central series of G. Another well known fact is that $[\Gamma^i G, \Gamma^j G] \subseteq \Gamma^{i+j} G$. In the sequel we shall also use the fact that Γ^i is generated by the commutators of length i (cf. [7] Theorem 10.2.1). We also recall the commutator identities: $[xy, z] = [x, z]^y [y, z]$, $[x, yz] = [x, z][x, y]^z$ and $x[x, y] = x^y$.

It must also be noted that we use the convention that commutators are written left-normed, namely $[x_1, x_2, \ldots, x_n] = [[\cdots[[x_1, x_2], x_3], \cdots], x_n]$.

LEMMA 3.6: *Let K be an arbitrary group. If $z \in \Gamma^\ell K$, then for all $y \in K$ and* *all* $x_1, \dots, x_n \in K$ *the following holds:*

$$[yz, x_1, \dots, x_n] = [y, x_1, \dots, x_n] \cdot [z, x_1, \dots, x_n] \pmod{\Gamma^{\ell+n+1}K}.$$

Proof: Use induction on n. For $n = 1$ we have

$$[yz, x_1] = [y, x_1]^z [z, x_1] = [y, x_1][y, x_1, z][z, x_1] = [y, x_1][z, x_1] \pmod{\Gamma^{\ell+2}K}.$$

Induction step for $n+1$: Denote

$$y^{(n)} = [y, x_1, \dots, x_n] \in \Gamma^{n+1}K,$$
$$z^{(n)} = [z, x_1, \dots, x_n] \in \Gamma^{\ell+n}K.$$

Use the induction hypothesis to compute:

$$\begin{aligned}[yz, x_1, \dots, x_{n+1}] &= [y^{(n)} \cdot z^{(n)}, x_{n+1}] \pmod{\Gamma^{\ell+n+2}} \\ &= [y^{(n)}, x_{n+1}]^{z^{(n)}} \cdot [z^{(n)}, x_{n+1}] \pmod{\Gamma^{\ell+n+2}} \\ &= [y^{(n)}, x_{n+1}] \cdot [z^{(n)}, x_{n+1}] \pmod{\Gamma^{\ell+n+2}}. \end{aligned}$$ ∎

PROPOSITION 3.7: *If K is nilpotent of class $c+1$, then for any choice of elements $x_1, \dots, x_c$ in K, and any choice of i, the mapping $y \mapsto [x_1, \dots, x_i, y, x_{i+1}, \dots, x_c]$ is a homomorphism $K \to Z(K)$.*

Proof: Let $u = [x_1, \dots, x_i]$. Observe that for every $y, z \in G$,

$$[u, yz] = [u, z][u, y][u, y, z].$$

The proof follows from the above lemma together with $\Gamma^{c+1}K = 1$. ∎

Proof of Theorem 3.5: Let $H = \mathrm{im}(a)$ and let $c = c(H)$. Clearly $c \le c(LG)$, and we prove that equality holds. By assumption $\Gamma^c H \neq 1$ and $\Gamma^{c+1}H = 1$. Assume to the contrary that $c(LG) > c(H)$, and let $k = c(LG)$. By assumption $\Gamma^k LG \neq 1$, but as $k > c$, $\Gamma^k H = 1$. For $x_1, \dots, x_{k-1} \in H$ we obtain a homomorphism φ: $LG \to Z(LG)$, φ: $y \mapsto [y, x_1, \dots, x_{k-1}]$, which satisfies $\varphi(H) = 1$, hence φ is trivial. It follows that

$$(1) \qquad [LG, \underbrace{H, \dots, H}_{k-1 \text{ times}}] = 1.$$

Now let $x_1 \in LG$ and $x_2, \dots, x_{k-1} \in H$, and define a map φ: $LG \to Z(LG)$ by the formula φ: $y \mapsto [x_1, y, x_2, \dots, x_{k-1}]$. Then φ is a homomorphism by the above proposition, and by (1) it satisfies $\varphi(H) = 1$. Hence φ is trivial and we obtain

$$[LG, LG, \underbrace{H, \dots, H}_{k-2 \text{ times}}] = 1.$$

Repeating this process $k-1$ times we finally conclude that $\Gamma^k LG = 1$, which is a contradiction. ∎

4. Words and nilpotency of G-modules

Recall the definition of a nilpotent G-module from [8]. Throughout this section IG denotes the augmentation ideal of the group-ring $\mathbb{Z}G$. We remark that the ideals $(IG)^n$ are generated as abelian groups by the elements

$$\{(g_1-1)\cdots(g_n-1)\colon g_1,\dots,g_n\in G\}\subseteq \mathbb{Z}G.$$

A G-module A is said to be nilpotent if $(IG)^nA=0$ for some $n\geq 0$. The nilpotency class of A (cf. [8]) is denoted by $c(A)$.

Given a group G and an ordinal λ one can construct a transfinite upper central series $\{Z_\alpha(G)\}_{\alpha\leq\lambda}$ by induction as follows. Begin with $Z_0(G)=1$. If α is a non-limit ordinal then define $Z_\alpha(G)$ by $Z_\alpha(G)/Z_{\alpha-1}(G)=Z(G/Z_{\alpha-1})$. If α is a limit ordinal then $Z_\alpha(G)=\bigcup_{\beta<\alpha}Z_\beta(G)$. We say that G has an exhaustive upper central series if there exists an ordinal λ such that $Z_\lambda(G)=G$.

Throughout this section (L,a) is a CIF in the category of G-modules (G is fixed). It should be noted that given a G-module A, then LA does *not* mean application of a CIF in the category $\mathcal{A}b$ to the underlying group of A and obtaining the G-structure by functoriality. For example, it is possible (as is the case in Example 4.2) that the underlying group of a G-module A is $\mathbb{Z}/p$ whereas the one of LA is $\mathbb{Z}/p\oplus\mathbb{Z}/p$, which is impossible for $L\colon \mathcal{A}b\to\mathcal{A}b$.

THEOREM 4.1: *Let A be a nilpotent G-module and assume that G has an exhaustive transfinite upper central series (e.g. G is nilpotent). Then LA is a nilpotent G-module, and $c(LA)\leq c(A)$.*

Proof: Let n be an integer such that $(IG)^nA=0$ and let λ be an ordinal such that $Z_\lambda(G)=G$. Our aim is to show that $(IG)^nLA=0$.

For every $g\in G$ let $\zeta(g)=\min\{\alpha\leq\lambda\colon g\in Z_\alpha(G)\}$. We abuse notation and write $\zeta(g_1,\dots,g_n)=(\zeta(g_1),\dots,\zeta(g_n))$ for the n-tuple $(g_1,\dots,g_n)\in G^n$. Well order λ^n lexicographically: $(\alpha_1,\dots,\alpha_n)\prec(\beta_1,\dots,\beta_n)$ if there exists $k\leq n$ such that $\alpha_i=\beta_i$ for all $i<k$ and $\alpha_k<\beta_k$.

In order to prove the theorem we must show that, for all $g_1,\dots,g_n\in G$,

$$(g_n-1)\cdots(g_1-1)LA=0. \tag{2}$$

We use induction on $\zeta(g_1,\dots,g_n)$. Notice that if $\zeta(g_1,\dots,g_n)=(0,\dots,0)$ then (2) holds trivially. Assume (2) holds whenever $\zeta(g_1,\dots,g_n)\prec(\zeta_1,\dots,\zeta_n)$. We prove it for $g_1,\dots,g_n\in G$ satisfying $\zeta(g_1,\dots,g_n)=(\zeta_1,\dots,\zeta_n)$.

Define (yet a map of abelian groups) $\Phi\colon LA\to LA$ by

$$\Phi(x)=(g_n-1)\cdots(g_1-1)x.$$

We show that Φ is a G-homomorphism. Namely, given $t \in G$ we must show that $\Phi(t \cdot x) = t \cdot \Phi(x)$ for all $x \in LA$. For convenience define $z_i = [g_i, t]$ and note that $\zeta(z_i) < \zeta(g_i) = \zeta_i$. Also observe that, for all $g \in G$,

$$\begin{aligned}(g-1)t &= t(g^t - 1),\\ g^t - 1 &= (g-1)([g,t]-1) + ([g,t]-1) + (g-1).\end{aligned}$$

Evidently $\Phi(tx) = (g_n - 1)\cdots(g_1 - 1)tx = t(g_n^t - 1)\cdots(g_1^t - 1)x$. Therefore, in order to show that Φ commutes with the action of t, it suffices to show that

$$(g_n^t - 1)\cdots(g_1^t - 1)x = (g_n - 1)\cdots(g_1 - 1)x \quad (x \in LA). \tag{3}$$

We now use induction to prove that

$$(g_n^t - 1)\cdots(g_1^t - 1)x = (g_n^t - 1)\cdots(g_{i+1}^t - 1)(g_i - 1)\cdots(g_1 - 1)x \quad (x \in LA). \tag{4}$$

If $i = 0$ then (4) is trivial. Assume (4) for $i = j - 1$ ($j > 0$); we prove it for $i = j$. Observe that

$$\begin{aligned}&\zeta(g_1, \dots, g_{j+1}, z_j, g_{j-1}, \dots, g_n) \prec (\zeta_1, \dots, \zeta_n),\\ &\zeta(g_1, \dots, g_{j+1}, z_j, g_j, \dots, g_{n-1}) \prec (\zeta_1, \dots, \zeta_n),\end{aligned}$$

so by induction hypothesis it follows that

$$\begin{aligned}&(g_n^t - 1)\cdots(g_j^t - 1)(g_{j-1} - 1)\cdots(g_1 - 1)x\\ &= (g_n^t - 1)\cdots((g_j - 1)(z_j - 1) - (z_j - 1) - (g_j - 1))(g_{j-1} - 1)\cdots(g_1 - 1)x\\ &= (g_n^t - 1)\cdots(g_{j+1}^t - 1)(g_j - 1)\cdots(g_1 - 1)x\end{aligned}$$

and the induction step is complete (for the proof of (4)).

Having proved (3) we deduce that Φ is a G-module homomorphism. Since $\Phi \circ a = 0$, where a is the coaugmentation, it follows that $\Phi = 0$. This establishes the induction step in the proof of (2). ∎

Example 4.2: The assumption that G has an exhaustive upper central series cannot be omitted. Let $V = (\mathbb{F}_p)^2$ be the two-dimensional vector space over the field $\mathbb{F}_p$, $p > 2$, and let $\{e_0, e_1\}$ be the standard base. Let $H = \mathrm{Aut}(V) = \mathrm{GL}_2(\mathbb{F}_p)$ and let

$$G = \mathrm{Stab}_H(e_0) = \left\{ g \in H \colon g = \begin{pmatrix} 1 & \alpha \\ 0 & \beta \end{pmatrix}, \alpha, \beta \in \mathbb{F}_p, \beta \neq 0 \right\}. \tag{5}$$

Let $U = \langle e_0 \rangle$ be the one-dimensional subspace spanned by e_0. View U and V as G-modules and observe that U is a trivial G-module. We now claim that the inclusion $j: U \to V$ localizes, i.e. it induces an isomorphism $j^*: \mathrm{Hom}_G(V, V) \to \mathrm{Hom}_G(U, V)$. To see this, we prove that both $\mathrm{Hom}_G(V, V)$ and $\mathrm{Hom}_G(U, V)$ consist of scalar transformations only. It is then immediate that j^* is bijective.

Indeed, let $\varphi \in \mathrm{Hom}_G(U, V)$. Clearly $\varphi(U) \subseteq U$ since U is the largest trivial submodule of V. Hence $\varphi = \lambda j$ for some $\lambda \in \mathbb{F}_p$.

Now, let $\varphi \in \mathrm{Hom}_G(V, V)$. By the argument above $\varphi(e_0) = \lambda e_0$ for some λ. Using the fact that φ commutes with the action of all $g \in G$, and using matrices g in the form given in (5) with $\alpha = 1$ and $\beta = \pm 1$ (recall that $p > 2$), it is easy to prove that $\varphi(e_1) = \lambda e_1$ so φ is indeed scalar.

Although U is a trivial module, V is not nilpotent. Observe that for $g \in G$ with $\alpha = 0, \beta = -1$ we have $(g-1)e_1 = -2e_1$, therefore $IG \cdot V \supset \langle e_1 \rangle$. It then follows that $(IG)^n V \supset \langle e_1 \rangle \neq 0$ for all n. This need not be surprising as one easily checks that $Z(G) = 1$.

A result closely related to that of Theorem 3.5 is the following proposition which is a special case of its succeeding one stating that in some cases the word problem has a positive answer in the category of G-modules.

PROPOSITION 4.3: *Let A be a nilpotent G-module, and assume that LA is also nilpotent. Then $c(LA) \leq c(A)$.*

Proof: Assume to the contrary that $n = c(LA) > c(A) = m$. Let $g_1, \ldots, g_{n-1}$ be elements of G and define a function (of sets) $\varphi: LA \to LA$ by

$$\varphi: x \mapsto (g_1 - 1) \cdots (g_{n-1} - 1) \cdot x.$$

For all $h \in G$ and $x \in LA$, $(h-1) \cdot \varphi(x) = \varphi((h-1)x) = 0$ because $c(LA) = n$. It follows, by the definition of φ, that $h \cdot \varphi(x) = \varphi(x) = \varphi(h \cdot x)$, i.e. φ is a G-homomorphism. However, $\varphi \circ a = 0$ where $a: A \to LA$ is the coaugmentation, hence $\varphi = 0$. In other words, $(IG)^{n-1} LA = 0$, a contradiction. ∎

THEOREM 4.4: *Let A be a nilpotent G-module such that LA is also nilpotent. If $w(x_1, \ldots, x_t)$ is a satisfiable word in A, then it is also satisfiable in LA.*

Proof: Every word is clearly equivalent to a word

$$w(x_1, \ldots, x_t) = r_1 x_1 + \cdots + r_t x_t \quad \text{where } r_i \in \mathbb{Z}G.$$

Replacing $x_j = 0$ for all $j \neq i$ we see that the words $r_i x_i$ are all satisfiable in A, and our problem reduces to showing that if the word $u(x) = r \cdot x$ ($r \in \mathbb{Z}G$)

is satisfiable in A then it is satisfiable in LA. Let $n = c(A)$. By the previous proposition, $c(LA) \leq n$.

CLAIM: *For all $g_1, \dots, g_m \in G$, $m \leq n$ and $i \geq 0$,*

$$[(g_1 - 1) \cdots (g_i - 1) \cdot r \cdot (g_{i+1} - 1) \cdots (g_m - 1)] \cdot LA = 0 \tag{6}$$

where if $i = 0$ or $i = m$ it is understood that there are no terms on the left or the right (respectively) of r.

Applying the claim for $m = 0$ we see that $r \cdot LA = 0$, which is exactly stating that the word $u(x) = r \cdot x$ is satisfiable in LA, as desired.

Proof of the claim: We use (descending) induction on m starting with $m = n$. In this case, $(g_1 - 1) \cdots (g_i - 1) \cdot r \cdot (g_{i+1} - 1) \cdots (g_m - 1) \in (IG)^n$, and therefore by the previous proposition (6) holds for $m = n$.

For the inductive step, assume that (6) holds for $1 \leq m + 1 \leq n$ and we prove it for m. Let $g_1, \dots, g_m$ be elements of G for which we wish to prove (6). Define a map $LA \to LA$ by

$$\varphi\colon x \mapsto (g_1 - 1) \cdots (g_i - 1) \cdot r \cdot (g_{i+1} - 1) \cdots (g_m - 1) \cdot x\ . \tag{7}$$

We wish to show φ is a G-homomorphism. Given $g \in G$, the induction hypothesis and (7) imply that $(g - 1)\varphi(x) = \varphi((g - 1)x) = 0$. This, in turn, implies that $\varphi(g \cdot x) = \varphi(x) = g \cdot \varphi(x)$, namely, φ is a G-homomorphism. Since $\varphi \circ a = 0$, where $a\colon A \to LA$ is the coaugmentation, it follows that $\varphi = 0$. In other words,

$$[(g_1 - 1) \cdots (g_i - 1) \cdot r \cdot (g_{i+1} - 1) \cdots (g_m - 1)] \cdot LA = 0$$

and the inductive step is complete. This concludes the proof of the claim and, consequently, the proof of the theorem. ∎

COROLLARY 4.5: *Let G have an exhaustive upper central series, and let A be a nilpotent G-module. If $w(x_1, \dots, x_t)$ is a word satisfiable in A, then it is also satisfiable in LA.* ∎

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